The Fauser Monoid

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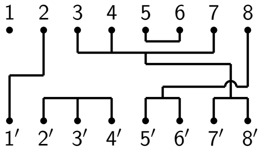
Diagrams

Let $k \in \mathbb{Z}^{>0}$, $K = \{1, ..., k\}$ and $K' = \{1', ..., k'\}$.
A **diagram** is a partition of $K \cup K'$.

Example

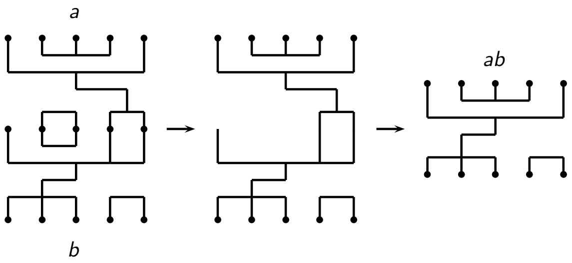
Take $k = 8$ and consider
$\{\{1\},\ \{2, 1'\},\ \{3, 4, 7, 7', 8'\},\ \{5, 6\},\ \{8, 5', 6'\},\ \{2', 3', 4'\}\}$.



- A diagram is **planar** if the edges can be drawn without crossing inside the rectangle bounding the vertices;
- **Transversal components** are edges that connect vertices in both rows;
- The **rank** of a diagram is the number of transversals it has.

Partition Monoid $\mathcal{P}_k$

Given two diagrams $a, b \in \mathcal{P}_k$, their product ab is formed pictorially as follows:



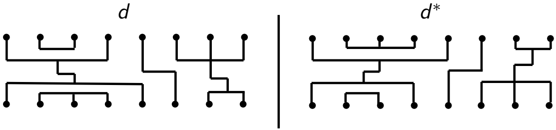
- place a on top of b;
- remove the middle dots and stranded loops; and
- clip loose ends and collapse remaining loops.

The monoid of diagrams under this product is called the **partition monoid** $\mathcal{P}_k$.

$\mathcal{P}_k$ is a regular *-semigroup

Pictorially we obtain d^* by flipping d upside down.

Example



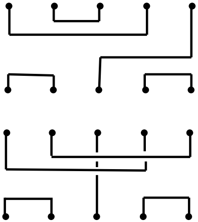
For each $d \in \mathcal{P}_k$:

- $d^{**} = d$ (* is an involution);
- $(de)^* = e^*d^*$ (* is an anti-homomorphism); and
- $dd^*d = d$ (regularity condition).

Jones Monoid $\mathcal{J}_k$ and Brauer Monoid $\mathcal{B}_k$

- **Jones Monoid** $\mathcal{J}_k$ consists of all planar matchings of $K \cup K'$;
- **Brauer Monoid** $\mathcal{B}_k$ consists of all matchings of $K \cup K'$.

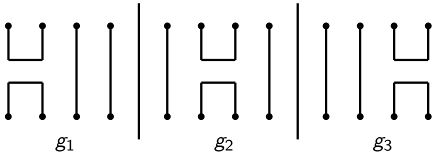
Examples



Diapsis Generators

Example

When $k = 4$ we have 3 **diapsis generators**:



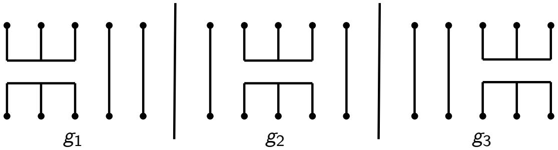
- $\mathcal{J}_k$ is generated by diapsis generators (and $\mathrm{id}_{\mathcal{P}_k}$);
- $\mathcal{B}_k$ is generated by diapsis generators and S_k.

Triapsis Generators

Consider what happens when we replace the diapses in the generators of $\mathcal{B}_k$ and $\mathcal{J}_k$ with triapses.

Example

When $k = 5$ we have 3 **triapsis generators**:

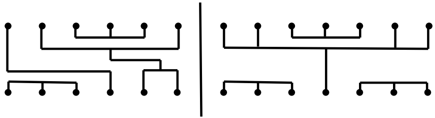


Obvious First Question: What diagrams are generated?

Triapsis Monoid $\mathcal{T}_k$ and Fauser Monoid $\mathcal{F}_k$

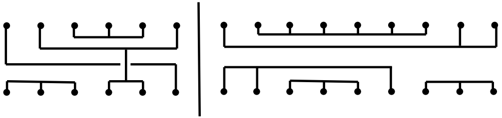
The **Triapsis Monoid** $\mathcal{T}_k$ consists of $\underline{\mathrm{id}_{\mathcal{P}_k}}$ and $d \in \mathcal{P}_k$ where:

- d is planar;
- there's at least one triapsis connecting consecutive points along the upper points, similarly along the lower points;
- for each edge $e \in d$, $|U(e)| \equiv |L(e)| \pmod 3$.



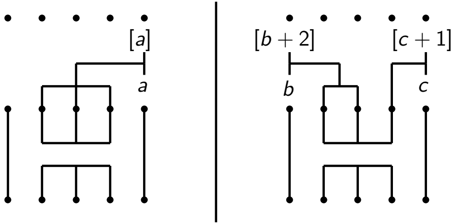
The **Fauser Monoid** $\mathcal{F}_k$ consists of $\underline{S_k}$ and $d \in \mathcal{P}_k$ where:

- there's at least one triapsis along the upper points, similarly along the lower points;
- for each edge $e \in d$, $|U(e)| \equiv |L(e)| \pmod 3$.



Characterising the elements of $\mathcal{T}_k$ ($\langle g_1, \ldots, g_{k-2}, \mathrm{id}_{\mathcal{P}_k} \rangle = \mathcal{T}_k$)

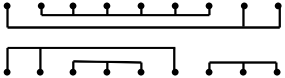
(⇒:) When showing $\langle g_1, \ldots, g_{k-2}, \mathrm{id}_{\mathcal{P}_k} \rangle \leq \mathcal{T}_k$, we show that $\mathcal{T}_k \langle g_1, \ldots, g_{k-2} \rangle \subseteq \mathcal{T}_k$.



(⇐:) To show that $\mathcal{T}_k \leq \langle g_1, \ldots, g_{k-2}, \mathrm{id}_{\mathcal{P}_k} \rangle$, we show how to decompose a diagram $d \in \mathcal{T}_k$ into a product of generators. We begin with $d = utl$, then decompose u, t and l.

Terminology Complications

- Triapsis Monoid is a better description of our generators than the elements of $\mathcal{T}_k$;
- Can't call $\mathcal{T}_k$ the planar version of $\mathcal{F}_k$; and



- I'm not overly fond of referring to $\mathcal{F}_k$ as the symmetric version of $\mathcal{T}_k$, plus if we can't think of descriptive names then we want to call $\mathcal{F}_k$ the Fauser monoid.

Why Fauser?

Cardinality of $\mathcal{F}_k$

Let

- $N(n, t_1, ..., t_k)$ be the number of ways to place t_1 triapses, ..., t_k $3k$-apses along n points;
- $N(n, t) = \Sigma_{3t_1+...+3kt_k=t:t_1>0} N(n, t_1, ..., t_k)$ be the number of ways to use t of n dots with non-transversals; and
- $T(n_1, n_2)$ be the number of ways to feasibly connect n_1 points to n_2 points with just (feasible) transversals.

We have the following recurrence relations for N and T:

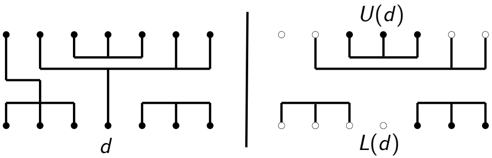
- $N(n, 0, ..., 0) = 1$;
- $N(n, t_1, ..., t_k) = N(n-1, t_1, ..., t_k) + \Sigma_{i:t_i>0} \binom{n-1}{3i-1} N(n-3i, t_1, ..., t_i - 1, ..., t_k)$;
- $T(0,0) = 1,\ T(n_1, n_2) = T(n_2, n_1),\ T(n_1, 0) = 0, n_1 > 0$;
- $T(n_1, n_2) = \Sigma_{(n_1', n_2') \leq (n_1, n_2): n_1' \equiv n_2' (\text{mod } 3)} \binom{n_1-1}{n_1'-1} \binom{n_2}{n_2'} T(n_1 - n_1', n_2 - n_2')$.

$$|\mathcal{F}_k| = n! + \Sigma_{u=1}^{\lfloor n/3 \rfloor} \Sigma_{l=1}^{\lfloor n/3 \rfloor} N(n, 3u) N(n, 3l) T(n-3u, n-3l).$$

Patterns

A **pattern** p is a partition of K (or K') with a two-tone vertex colouring, where the colour of a vertex indicates whether the edge connected to it is a transversal or non-transversal component.

Example



Hence we can break a diagram $d \in \mathcal{P}_k$ into its **upper pattern** $U(d)$ and **lower pattern** $L(d)$.

$\mathcal{T}_k$-admissibility and $\mathcal{T}_k$-compatibility

- a pattern p is $\mathcal{T}_k$-**admissible** if $\exists\ d \in \mathcal{T}_k$ with $U(d) = p$; and
- $\mathcal{T}_k$-admissible p, q are $\mathcal{T}_k$-**compatible** if $\exists\ d \in \mathcal{T}_k$ with $U(d) = p$ and $L(d) = q$.
  (d is unique, which we denote by $\delta(p, q)$)

Properties

- $\mathcal{T}_k$-compatibility is an equivalence relation; and
- $\delta(p, q)\delta(q, r) = \delta(p, r)$.

Characterising $\mathcal{T}_k$-admissibility and $\mathcal{T}_k$-compatibility

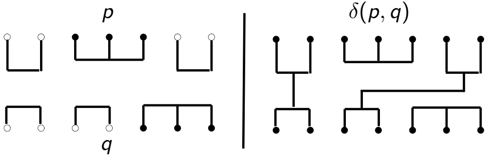
A pattern p is $\mathcal{T}_k$-admissible iff:

- it is planar;
- it has at least one triapsis; and
- each non-transversal component has cardinality divisible by 3.

$\mathcal{T}_k$-admissible patterns p, q are $\mathcal{T}_k$-compatible iff:

- $rank(p) = rank(q)$; and
- the cardinalities of *matched* transversal components are congruent mod 3.

Example



Green's Relations

Definition

For $a, b \in S$:

- $\mathcal{R} = \{(a, b) \in S^2 : aS^1 = bS^1\}$;
- $\mathcal{L} = \{(a, b) \in S^2 : S^1 a = S^1 b\}$;
- $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$; and
- $\mathcal{J} = \{(a, b) \in S^2 : S^1 a S^1 = S^1 b S^1\}$;

Theorem (Howie)

If $T \leq S$ is regular then Green's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ relations are just their respective restrictions on T. Ie. $\mathcal{L}^T = \mathcal{L}^S \cap T^2$.

Green's Relations on $\mathcal{P}_k$

Theorem (Wilcox)

For $a, b \in \mathcal{P}_k$:

- $a\mathcal{R}b$ iff $U(a) = U(b)$;
- $a\mathcal{L}b$ iff $L(a) = L(b)$;
- $a\mathcal{H}b$ iff $U(a) = U(b)$ and $L(a) = L(b)$; and
- $a\mathcal{J}b$ iff $rank(a) = rank(b)$.

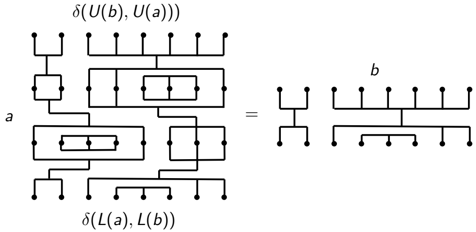
Green's $\mathcal{J}$ Relation on $\mathcal{T}_k$

Theorem

For $a, b \in \mathcal{T}_k$, $a\mathcal{J}b$ iff '$U(a)$ and $U(b)$ are $\mathcal{T}_k$-compatible'.

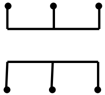
$\underline{(\Rightarrow:)}$ 'rank(a) = rank(ab)' $\Rightarrow$ '$U(a)$ and $U(ab)$ are $\mathcal{T}_k$-compatible'.

$$\begin{aligned}\underline{(\Leftarrow:)}[\delta(U(b), U(a)).a].\delta(L(a), L(b)) &= \delta(U(b), L(a)).\delta(L(a), L(b)) \\ &= \delta(U(b), L(b)) \\ &= b\end{aligned}$$

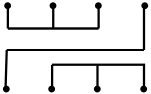


Presentation of $\mathcal{T}_k$

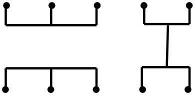
$a^2 = a$



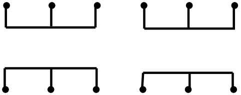
$aba = a$
$(bab = b)$



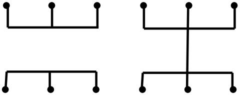
$abca = aca = acba$



$ad = da$



$abdca = acdba$



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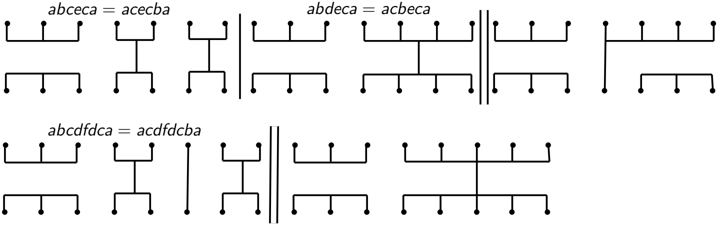
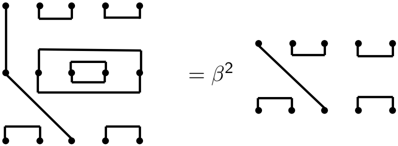


Diagram Algebras

- For $\beta \in \mathbb{C}$, the **partition algebra** $\mathbb{C}^\beta[\mathcal{P}_k]$ is the semigroup algebra $\mathbb{C}[\mathcal{P}_k]$ with multiplication $d * d' = \beta^r(dd')$ where r is the number of blocks removed when forming dd'.



- $\mathbb{C}^\beta[\mathcal{B}_k]$ and $\mathbb{C}^\beta[\mathcal{F}_k]$ are defined analogously.

Schur-Weyl Duality

Schur-Weyl duality tells us (amongst other things) that:

- $\mathrm{GL}_n(\mathbb{C})$ and $\mathbb{C}[S_k]$ have commuting actions on $V^{\otimes k}$;
- Each action generates the full centraliser of the other Ie. $\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}(V^{\otimes k}) = \mathbb{C}[S_k]$ and $\mathrm{End}_{\mathbb{C}[S_k]}(V^{\otimes k}) = \mathrm{GL}_n(\mathbb{C})$.

A number of analogous dualities are known, for example between:

- $\mathrm{O}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ and $\mathbb{C}^{\beta}[\mathcal{B}_k] \supseteq \mathbb{C}[S_k]$ (Brauer);
- $S_n \subseteq \mathrm{O}_n$ and $\mathbb{C}^{\beta}[\mathcal{P}_k] \supseteq \mathbb{C}^{\beta}[\mathcal{B}_k]$ (Martin);
- IS_n and $\mathbb{C}[I_k^*]$ (Kudryavtseva & Mazorchuk).

We are hoping to find a subgroup of $GL_n(\mathbb{C})$ which is in a Schur-Weyl type duality with $\mathbb{C}^{\beta}[\mathcal{F}_k]$.

Suggestions and/or Questions?